On Levels in Arrangements of Lines, Segments, Planes, and Triangles*

Pankaj K. Agarwal† Boris Aronov‡ Micha Sharir§

Abstract

We consider the problem of bounding the complexity of the k-th level in an arrangement of n curves or surfaces, a problem dual to, and extending, the well-known *k-set problem.* (a) We review and simplify some old proofs in new disguise and give new proofs of the bound $O(n\sqrt{k+1})$ for the complexity of the k-th level in an arrangement of n lines. (b) We derive an improved version of Lovász Lemma in any dimension, and use it to prove a new bound, $O(n^2k^{2/3})$, on the complexity of the k-th level in an arrangement of n planes in $\mathbb{R}^3$, or on the number of k-sets in a set of n points in three dimensions. (c) We show that the complexity of any single level in an arrangement of n line segments in the plane is $O(n^{3/2})$, and that the complexity of any single level in an arrangement of n triangles in 3-space is $O(n^{17/6})$.

1 Introduction

Background. The *k-set problem* is one of the most challenging open problems in combinatorial geometry. The simplest variant of the problem is: Given a set S of n points in the plane in general position, and a parameter $0 \le k \le n-2$, what is the maximum possible number of lines that pass through a pair of points of S and have exactly k points of S in one of the open halfplanes that they define? In a dual setting, we are given a set $\mathcal{L}$ of n lines in the plane in general position, and want to bound the maximum possible number of vertices v of the arrangement $\mathcal{A}(\mathcal{L})$, such that exactly k lines pass below v. We denote this set of vertices by V_k. (Strictly speaking, $|V_k|$ is a slightly different quantity than the one defined above, as it corresponds to the number of lines passing through two of the given points and having exactly k of the remaining points *below* them.) A slightly different variant of the dual problem is to define the *k-th level* of the arrangement, as the closure of the set of points that lie on the lines and have exactly k lines below them, and seek a bound on the number of vertices of that level. (Each vertex of this closure may have either k or $k-1$ lines below it.) See Figure 1 for an illustration.

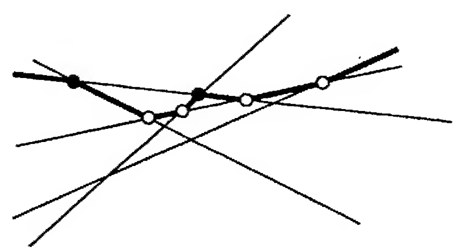

Figure 1: The third level in an arrangement of lines. The vertices of V_2 are indicated by empty circles and the vertices of V_3 are marked by black circles.

The k-set problem was first studied about 1970 by Erdős et al. and by Lovász [11, 19]. These papers have established an upper bound $O(n\sqrt{k+1})$ and a lower bound $\Omega(n\log(k+1))$ on the desired quantity, leaving a fairly big gap that is still mostly open. The only further progress on this problem is due to Pach et al. [21], where the upper bound is slightly improved to $O(n\sqrt{k+1}/\log^*(k+1))$; see [4, 10] for related results.

In the dual setting, the problem can be generalized in an obvious manner: In the plane, we are given a collection Σ of n x-monotone curves, each being the graph of a continuous totally or partially defined function, and a parameter $0 \le k < n$, and wish to bound the *complexity* (i.e., the number of vertices) of the k-th level in the arrangement $\mathcal{A}(\Sigma)$, defined exactly as in the case of lines. In this more general

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†Department of Computer Science, Box 90129, Duke University, Durham, NC 27708-0129, USA

‡Department of Computer and Information Science, Polytechnic University, Brooklyn, NY 11201-3840, USA

§School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel and Courant Institute of Mathematical Sciences, New York University, New York, NY 10012, USA

setting only two results are known: A recent seemingly weak, but elegant analysis by Tamaki and Tokuyama [23] yields the bound $O(n^{23/12})$ on the complexity of a level in an arrangement of n *pseudo-parabolas*, which are graphs of total functions, each pair of which intersect at most twice. We also mention the case of *pseudo-lines*, which are x-monotone connected curves, unbounded at either end, each pair of which intersect exactly once, where a slightly larger lower bound of $\Omega(n \cdot 2^{c\sqrt{\log n}})$ for the complexity of the median level is established in [18]. Our proof techniques and upper bounds apply equally well to the case of pseudo-lines.

Similar extensions apply in higher dimensions. In the primal setting, we are given a set S of n points in $\mathbb{R}^d$ in general position, and wish to bound the number of hyperplanes passing through d of the points such that one of the halfspaces that they bound contains exactly k points of S. For $d = 3$, the best known upper and lower bounds are, respectively, $O(n^{8/3})$ and $\Omega(n^2 \log n)$ [3, 8]. For $d > 3$, the best known upper bound is $O(n^{d-c_d})$, for some exponentially small but positive constant c_d [26]. Note that, in contrast to the planar case, these bounds depend only on n and not on k.

We can formulate the problem in general dimension, in a dual setting: We consider an arrangement of hyperplanes, or, more generally, of surfaces that are graphs of continuous total or partial functions, and define the k-th level of the arrangement exactly as in the planar case. We now seek bounds on the maximum possible number of vertices (or of faces of all dimensions) of the level. Except for the case of hyperplanes, which is equivalent to the k-set problem mentioned in the preceding paragraph, no nontrivial bounds for the entire range of values of k are known.

In spite of the sorry state of the problem, one can obtain nontrivial bounds when k is small. The probabilistic analysis of Clarkson and Shor [7] (see also [22]) yields fairly sharp bounds on the combined complexity of the first k levels in arrangements. For the case of hyperplanes, for example, the bound is $\Theta(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil})$. For sufficiently small k, this gives a better upper bound on the complexity of a single level than the general bound stated above. The analysis of Clarkson and Shor [7] also implies, under fairly general assumptions that, for a constant $k > 0$, the worst-case number of vertices of the k-th level is asymptotically proportional to the maximum possible number of vertices on the lower envelope (i.e., the 0-th level) of the surfaces.

New Results. In this paper we make several contributions to these problems:

We first review some old proofs in new disguise, and present new proofs of the upper bound $O(n\sqrt{k+1})$ for the original planar k-set problem (or, dually, for the case of the k-th level in an arrangement of n straight lines in the plane). We review the proof technique of Gusfield [13], which, as we perceive, is not well known within the combinatorial and computational geometry communities, and show its relationship to other proofs. We also give a simple proof of the dual version of what we call "Lovász Lemma" (see Lemma 2.3 below) that is used to prove the bound. As is well known, these techniques apply equally well to arrangements of pseudo-lines; see, for example, [12].

We adapt two of our proof techniques to yield the bound $O(n^{3/2})$ on the complexity of a single level in an arrangement of n line segments (or "pseudo-segments," to be defined below). As far as we know, this bound is new.

We then proceed to study the problem in higher dimensions. First we obtain an improved version of Lovász Lemma that has a rather simple proof. Specifically, we show that no line can intersect more than $O(k^{d-1})$ k-set simplices, where a *k-set simplex* is a $(d-1)$-dimensional simplex, spanned by d points of S, such that the hyperplane containing the simplex has exactly k points of S in one of its open halfspaces. The previous bound was $O(n^{d-1})$ (see [3, 19]), so this is a significant improvement when $k \ll n$. Plugging the new bound into the analysis technique of [8], we show that the complexity of the k-th level in an arrangement of n planes in 3-space, and thus, the number of k-sets in a set of n points in $\mathbb{R}^3$, is $O(n^2 k^{2/3})$. This is the first general bound for arrangements of planes that depends on k, besides the aforementioned $O(nk^2)$ bound on the overall complexity of the first k levels. The new bound is an improvement when $k = \Omega(n^{3/4})$. A similar improved bound, of the form $O(n^{d-c_d-\varepsilon_d} k^{\varepsilon_d})$, can be obtained in any dimension $d > 3$, for appropriate constants ε_d, c_d, depending only on d, by combining the strengthened Lovász Lemma with the analysis of Živaljević and Vrećica [26]; see also [3].

Finally, we consider the case of triangles in 3-space, and show that the complexity of a single level in an arrangement of n such triangles is $O(n^{17/6})$.

2 Arrangements of Lines

Let $\mathcal{L}$ be a collection of n lines in the plane in general position. Let V_k, for $k = 0, \ldots, n-2$, denote, as in the introduction, the set of vertices of the arrangement $\mathcal{A}(\mathcal{L})$ that have exactly k lines below them. Then the set of vertices of the k-th level is $V_k \cup V_{k-1}$ (or just V_k, for $k = 0$). When the level passes through a vertex of V_{k-1} (resp. of V_k), it bends to the left (resp. to the right) as we traverse it in the positive x-direction. See Figure 1.

In this section we give four proofs of the following well-known result (which, as already pointed out, provides an estimate which is slightly larger than the best currently known upper bound of Pach *et al.* [21]):

Theorem 2.1 *The complexity of the k-th level of $\mathcal{A}(\mathcal{L})$ is $O(n\sqrt{k+1})$.*

Remark: In most of the following proofs we will actually argue that $|V_{k-1}| = O(n\sqrt{k})$. The claimed bound on the number of vertices of the k-th level follows by repeating the argument for $|V_k|$ and combining the two estimates.

First Proof (Potential Function): This proof is not new, and is an adaptation of the analysis technique of Gusfield [13, 14]. We give it for the sake of completeness, and because we will shortly apply a variant of it to the case of segments. We note that the way it is presented below is somewhat different than Gusfield's own analysis; we will further comment on Gusfield's analysis later on.

Let the lines in $\mathcal{L}$ be $\ell_1, \ell_2, \ldots, \ell_n$, sorted in the order of decreasing slope, and let k denote the given level. For any $\alpha \in \mathbb{R}$, we say that the level of a line $\ell \in \mathcal{L}$ is j at α if exactly j lines of $\mathcal{L}$ intersect the vertical line $x = \alpha$ below ℓ. For each $x \in \mathbb{R}$, define the *potential function*

$$\Phi(x) = \sum \{ j \mid \text{ the level of } \ell_j \text{ at } x \text{ is } < k \}.$$

We clearly have $\Phi(-\infty)$, $\Phi(+\infty) = O(nk)$ (in fact, $\Phi(x) = O(nk)$ for each x). As we sweep $\mathcal{A}(\mathcal{L})$ with a vertical line from left to right, the value of $\Phi(x)$ can change only when x equals the abscissa v_x of a vertex $v \in V_{k-1}$ (refer to Figure 1; note that $\Phi(x)$ does not change at vertices of V_k.) Suppose

that $v \in V_{k-1}$ is the intersection of lines ℓ_i and ℓ_j, with $j > i$. Then, as easily checked, the change $\Delta\Phi(v_x) = \Phi(v_x + \varepsilon) - \Phi(v_x - \varepsilon)$, for a sufficiently small $\varepsilon > 0$, is $j - i > 0$. In other words

$$\Phi(+\infty) = \Phi(-\infty) + \sum_{v \in V_{k-1}} \Delta\Phi(v_x) = O(nk),$$

with each of these changes being a positive integer.

The number of vertices v at which $\Delta\Phi(v_x) > \sqrt{k}$ is no more than $O(n\sqrt{k})$, as the sum of $\Delta\Phi(v_x)$ at these vertices is $O(nk)$, and each term in the sum is larger than $\sqrt{k}$. Concerning v at which the change is at most $\sqrt{k}$, there are at most $n-1$ vertices with corresponding pairs of indices $(i, i+1)$, $n-2$ vertices with pairs $(i, i+2)$, etc., for a total of

$$(n-1) + (n-2) + \cdots + (n - \sqrt{k} + 1) < n\sqrt{k}$$

vertices. Combining the two estimates, we conclude that $|V_{k-1}| = O(n\sqrt{k})$. In fact, a more careful counting gives the bound $2n\sqrt{k}$. □

Second Proof (Concave Chains): Let V_{k-1} denote, as above, the set of all vertices of the k-th level of $\mathcal{A}(\mathcal{L})$ at which the level makes a left turn, passing from a line with smaller slope to a line with larger slope. We associate with the k-th level a collection of k *concave chains*, where each such chain is an unbounded x-monotone concave polygonal curve contained in the union of the lines of $\mathcal{L}$. (As will be seen below, the chains in a certain sense "cover" the portion of $\mathcal{A}(\mathcal{L})$ below the level.) This is done as follows. The desired chains, denoted $c_1, \ldots, c_k$, start at $x = -\infty$ along the k lowest lines of the arrangement (these are the lines with the k largest slopes). Whenever some chain c_i reaches the k-th level, we are at a vertex $v \in V_{k-1}$, as is easily checked. We then continue c_i to the right along the other line incident to v. The chains bend only at vertices of V_{k-1}; otherwise each chain follows the line it is on. See Figure 2 for an illustration.

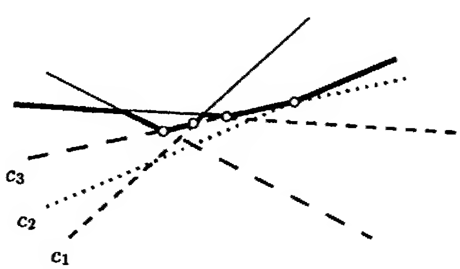

Figure 2: The concave chains associated with the third level; the level itself is drawn in bold, and the dashed paths denote the concave chains c_1, c_2, c_3.

It is easily seen that the resulting chains satisfy the following properties:

(i) The union of the chains is the closure of the portion of the union of the lines that lies below the k-th level. Except for the vertices of V_{k-1}, the union of the chains lies strictly below the k-th level.

(ii) The chains are vertex-disjoint and have non-overlapping edges, but they generally cross each other.

(iii) All the vertices of the chains lie on the upper envelope of the chains. Indeed, each chain, except for its vertices, lies fully below the k-th level, so any vertex of any chain lies above all the chains that are not incident to it.

Gusfield's analysis, with minor modifications, essentially establishes the following more general bound. The same result has been obtained independently by Halperin and Sharir [15], who were not aware of Gusfield's earlier work:

Theorem 2.2 ([13, 14, 15]) *The overall number of vertices of k concave chains, which are vertex-disjoint and have non-overlapping edges, in an arrangement of n lines in the plane, is $O(n\sqrt{k})$.*

This result clearly yields another proof Theorem 2.1. We leave it to the reader to verify that the potential-function proof applies, almost verbatim, to the general case of concave chains. □

Note that the above bound does not count crossings between chains.

Remarks: (1) If the concave chains are not allowed to cross each other, then their overall complexity is only $O(k^{2/3}n^{2/3} + n)$, as shown in [15, 16], but the analysis in these papers crucially relies on the fact that the chains cannot cross.

(2) The chains associated with the k-th level have the additional property (iii) that all their vertices appear on their upper envelope. Can a sharper upper bound be proved for the complexity of k concave chains with this extra property? Note, however, that property (iii) is not strong enough to improve the bound beyond $O(k^{2/3}n^{2/3} + n)$, as it is possible to produce a collection of k chains in an arrangement of n lines with a total of $\Omega(k^{2/3}n^{2/3} + n)$ vertices, so that all these vertices occur on the combined upper envelope of the chains [17].

Third Proof (Concave Chains and Lovász Lemma): One of the standard (and among the first) ways to prove the theorem is via the following statement, which we will refer to as "Lovász Lemma" [19]. It is usually stated in the primal plane, for a collection of n points, but we will state a dual version of the lemma, for arrangements of lines, and give a simple proof that uses the concave chain structure, with the aim of extending it to other types of arrangements.

Let $\mathcal{L}$ be a collection of n lines in the plane in general position, and let $1 \le k \le n-1$. For each vertex $v \in V_{k-1}$, let W_v denote the double wedge formed by the two lines that meet at v (W_v is the region between the upper and lower envelopes of these two lines). See Figure 3.

Lemma 2.3 (Dual Lovász Lemma in 2D) *For any point z in the plane not lying on any line of $\mathcal{L}$, the number of double wedges W_v, for $v \in V_{k-1}$, that contain z is at most* $2\min\{k, j\} \le 2k$, *where j is the number of lines of $\mathcal{L}$ that pass below z. Actually, there are at most* $\min\{k, j\}$ *left wedges and at most* $\min\{k, j\}$ *right wedges that contain z.*

Proof: Let c be one of the concave chains obtained in the previous proof, and consider the system of double wedges W_v, over all vertices v of c. The concavity of c is easily seen to imply that z can lie in at most two of these double wedges (in at most one right wedge and in at most one left wedge). Since we have k such chains, z can lie in at most $2k$ double wedges W_v, for $v \in V_{k-1}$. Moreover, if z lies above exactly

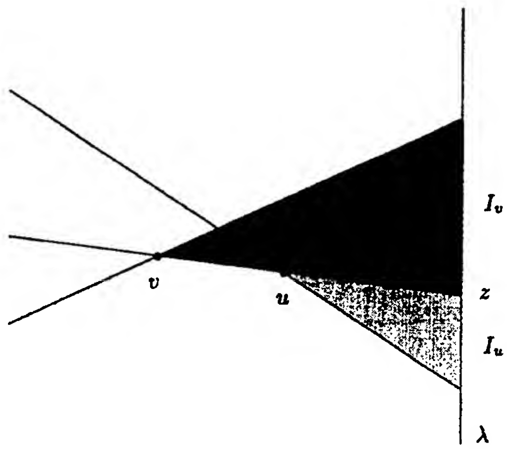

Figure 3: The setup for Lovász Lemma for the case of lines. The right wedges of W_v, W_u are shown shaded.

$j < k$ lines of $\mathcal{L}$, then it lies only above j concave chains, and can therefore only belong to double wedges corresponding to vertices of these chains. This easily implies the lemma. $\square$

We can now complete the third proof of the theorem, using an analysis dual to that in the original proof in [19]. That is, fix a vertical line λ, and intersect each W_v, for $v \in V_{k-1}$, with λ, to obtain a system of $|V_{k-1}|$ intervals on λ, having a total of n endpoints (which are the intersections of the lines in $\mathcal{L}$ with λ). It follows by a simple (and standard) counting argument (such as in [5, 19]) that λ must contain a point that lies in at least $|V_{k-1}|^2/4n^2$ intervals. Since this number cannot be more than $2k$, we obtain $|V_{k-1}| \le 2\sqrt{2}n\sqrt{k}$. With some care, this can be improved to $2n\sqrt{k}$.$\square$

Fourth Proof (Concave Chains and Cauchy-Schwarz Inequality): Let $\mathcal{L} = \{\ell_1, \ldots, \ell_n\}$ be a set of n lines, k be an integer between 0 and $n-2$, and $c_1, \ldots, c_k$ be the concave chains associated with the k-th level of $\mathcal{A}(\mathcal{L})$, as defined above.

Let w_{ij} denote the number of lines common to chains c_i and c_j, for $1 \le i < j \le k$. For each $q = 1, \ldots, n$, let M_q denote the number of chains that have an edge contained in ℓ_q. Note that $|V_{k-1}| = \sum_{q=1}^{n} M_q - k$. On the other hand, we have

$$\sum_{i,j} w_{ij} = \sum_{q=1}^{n} \binom{M_q}{2}.$$

Hence,

$$\begin{aligned} |V_{k-1}| &= \sum_{q=1}^{n} M_q - k = \sum_{q=1}^{n} (M_q - 1) + n - k \\ &\le (2n)^{1/2} \left[\sum_{q=1}^{n} \binom{M_q}{2} \right]^{1/2} + n - k \\ &= (2n)^{1/2} \left(\sum_{i,j} w_{ij} \right)^{1/2} + n - k. \end{aligned}$$

Fix a pair of chains c_i, c_j. The concavity of the chains is easily seen to imply that both c_i and c_j must lie on or below each of the w_{ij} common lines. That is, both c_i and c_j lie below (or on) the lower envelope E of these lines, so each chain touches each of these lines only at its unique segment that appears on the lower envelope. It follows that c_i and c_j must intersect each other at least once below each of the w_{ij} segments of E, so t_{ij}, the number of crossings between c_i and c_j, must be at least w_{ij}. See Figure 4. We thus have $\sum_{i,j} w_{ij} \le \sum_{i,j} t_{ij} \le nk$, where the latter inequality follows from the fact that the number of chain-crossings is equal to the number of vertices of $\mathcal{A}(\mathcal{L})$ at level $< k$, and this number is known to be at most nk [1]. This implies that

$$\begin{aligned} |V_{k-1}| &\le (2n)^{1/2} \left(\sum_{i,j} w_{ij} \right)^{1/2} + n - k \\ &\le (2n)^{1/2} (nk)^{1/2} + n - k \\ &= 2^{1/2} n\sqrt{k} + n - k. \end{aligned}$$

Note that the constant of proportionality is better than those yielded by the earlier proofs. $\square$

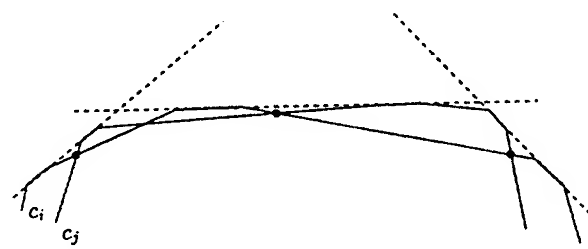

Figure 4: The chains c_i and c_j must cross at least w_{ij} times.

3 Arrangements of Segments

In this section we extend some of the proofs given above to the case of line segments. Let S be a collection of n segments in the plane in general position. For $k = 0, \ldots, n-1$, the k-th level in the arrangement $\mathcal{A}(S)$ of S is defined, as in the case of lines, to be the closure of the set of all points w that lie on segments of S and are such that the open downward-directed vertical ray emanating from w intersects exactly k segments of S (that is, there are k segments of S below w). The *complexity* of a level is the number of vertices of $\mathcal{A}(S)$ that lie on the level plus the number of discontinuities of the level. (Unlike the case of lines, a level of $\mathcal{A}(S)$ is not necessarily connected, and it may involve vertical jumps from a segment to the segment lying directly above or below it, when a new segment starts or ends at a point below the level. Clearly, the number of such discontinuities is at most $2n$.) As in the case of lines, we define V_k, for $k = 0, \ldots, n-2$, to be the set of vertices of $\mathcal{A}(S)$ (excluding segment endpoints) that have exactly k segments passing below them. The set of vertices of the k-th level, excluding segment endpoints and jump discontinuities, is $V_{k-1} \cup V_k$. The level bends to the left at vertices of V_{k-1} and to the right at vertices of V_k. See Figure 5 for an illustration.

Theorem 3.1 *The complexity of any single level in an arrangement of n line segments in the plane in general position is $O(n^{3/2})$.*

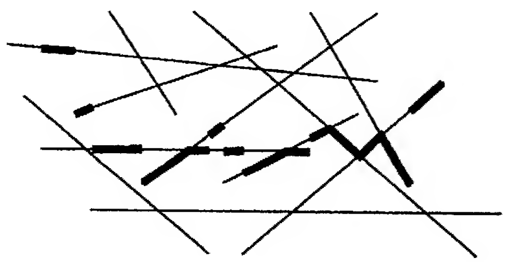

Figure 5: The second level in an arrangement of segments; here $|V_1| = 1$ and $|V_2| = 4$.

First Proof (Potential Function): This proof is an adaptation of the potential function proof for the case of lines, as given above. Let the segments be $s_1, s_2, \ldots, s_n$, sorted in the order of decreasing slope of their containing lines, and let k denote the given level. For each x, define the potential function

$$\Phi(x) = \sum \{ j \mid s_j \text{ lies at } x \text{ at level } < k \} .$$

Assuming that all the given segments are bounded, we have $\Phi(-\infty) = \Phi(+\infty) = 0$, and in any case we have, as above, $\Phi(x) = O(nk)$ for any x. As we sweep $\mathcal{A}(S)$ from left to right, we are interested in the signed changes $\Delta\Phi(x)$ in $\Phi(x)$. The value of $\Phi(x)$ can change only in one of the following three cases:

(i) x is the abscissa of the left endpoint of some segment s_i, and this endpoint lies below the current k-th level: In this case we have $\Delta\Phi(x) = i - j$, where s_j is the segment that currently lies directly below the level. (Here we have a discontinuity, where the level jumps down one segment.) Note that we may have $i = j$.

(ii) x is the abscissa of the right endpoint of some segment s_i, and this endpoint lies on or below the current k-th level: In this case we have $\Delta\Phi(x) = j - i$, where s_j is the segment that currently lies on the level. (Here we have a discontinuity, where the level jumps up one segment.) Again, it is possible that $i = j$.

(iii) x is the abscissa of a vertex $v \in V_{k-1}$ (as in the case of lines, $\Phi(x)$ does not change at vertices of V_k): Suppose that v is the intersection of segments s_i and s_j, with $j > i$. Then $\Delta\Phi(x) = j - i > 0$.

The number of events of types (i) and (ii) is at most $2n$, and the change in the potential at each of these events has absolute value $O(n)$, for a total change of absolute value $O(n^2)$. We thus have

$$\sum_x \Delta\Phi(x) = O(n^2) ,$$

where the summation is taken over all x that are the abscissae of a vertex of V_{k-1}, and each of these changes is a positive integer.

The proof now proceeds exactly as in the first proof of Theorem 2.1, and we leave it to the reader to fill in the straightforward details. The difference between the cases of segments and of lines is that, in the case of segments, we can only bound the total change in potential by $O(n^2)$, rather than by $O(nk)$. In fact, in the case of segments, the bound $O(n\sqrt{k+1})$ is too small for small values of k. For example, for $k = 0$ (that is, for the lower envelope of the segments) the complexity of the level can be $\Omega(n\alpha(n))$ [25], which is larger than the above bound. On the other hand, the complexity of the k-th level is smaller than the overall complexity of the first k levels, which is $O(n(k+1)\alpha(n/(k+1)))$ [22]. This is a better upper bound for small values of k. □

Second Proof (Concave Chains and Lovász Lemma): We next present a second proof, based on a variant of the dual Lovász Lemma given above. We use the same setup as above. That is, for each $v \in V_{k-1}$, we define the double wedge W_v formed by the two lines containing the segments incident to v.

Lemma 3.2 *For any point $z \in \mathbb{R}^2$, not lying on any line containing a segment of S, the number of double wedges W_v that contain z is at most $4n$.*

Proof: Let us first extend the notion of concave chains to the case of segments. The chains are constructed as follows. We start a new chain at (i) the left endpoint of any segment, if that endpoint lies below the k-th level, and (ii) at any point of discontinuity of the level, when the level jumps *up* from a segment s_i to a segment s_j (the chain is started along the lower segment s_i). As x increases, each chain c follows the segment that it lies on, except when of the following situations occurs:

(i) c reaches the right endpoint of that segment, and then c terminates there;

(ii) c follows a segment s_i and reaches a discontinuity of the k-th level, where the level jumps *down* to s_i, in which case c is terminated at that point; or

(iii) c reaches a vertex $v \in V_{k-1}$, in which case c bends to the right, and continues along the other segment incident to v.

We thus get a collection of at most $2n$ concave chains. It is easy to verify that these chains also satisfy (appropriate variants of) properties (i)–(iii) in the second proof of Theorem 2.1. Here the chains are graphs of partially-defined functions. Note that the domain of definition of some chains may also include intervals over which the k-th level is not defined (because there are fewer than $k+1$ segments over such an interval).

The proof can now be completed as in the case of lines, because, for each of the at most $2n$ chains c, a point z can belong to at most two double wedges W_v, for $v \in V_{k-1} \cap c$. □

The proof of Theorem 3.1 now proceeds along the same lines as the third proof of Theorem 2.1.

Remarks: (1) Both proofs presented in this section also apply to the cases of *pseudo-lines* and *pseudo-segments*. We have already defined the notion of a family of pseudo-lines. A collection S of n x-monotone connected arcs is *a family of pseudo-segments* if each of them can be extended to an x-monotone connected unbounded curve, so that this family of curves is a collection of pseudo-lines. (This is a much stronger definition than just requiring each pair of pseudo-segments to intersect at most once; see Figure 6.) We leave it to the reader to verify that both proofs go through in the case of pseudo-segments, with straightforward modifications.

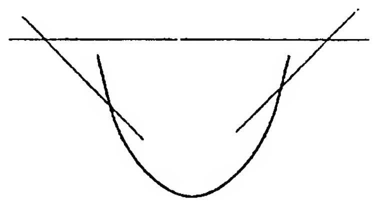

Figure 6: These five arcs do not form an arrangement of pseudo-segments.

(2) The immediate challenge is to improve Theorem 3.1, and obtain a better upper bound that also depends on k. As noted above, such a bound cannot be $O(n\sqrt{k+1})$, at least for small values of k.

(3) Another interesting open problem is to obtain an improved bound for the complexity of a single level in the arrangement of q piecewise-linear functions, whose graphs consist of a total of n segments. Of course, the bound $O(n^{3/2})$ applies here, but perhaps one can obtain an improved bound that is also a function of q and is smaller when $q \ll n$.

4 Improved Lovász Lemma and Arrangements of Planes

Let $P = \{\pi_1, \ldots, \pi_n\}$ be a collection of n planes in 3-space in general position, and let $\mathcal{A}(P)$ denote the arrangement of P. The k-th level of $\mathcal{A}(P)$ is defined as the closure of the set of all points that lie in the union of the planes and have exactly k planes lying below them. The complexity of the level, regarded as a polyhedral surface, is the number of its vertices, edges and faces. This is clearly proportional to only the number of vertices, and we will focus on bounding this latter quantity.

Theorem 4.1 *The number of vertices of the k-th level of $\mathcal{A}(P)$ is $O(n^2k^{2/3})$.*

As already noted, this improves the bound $O(n^{8/3})$ that was established in [8], when $k \ll n$, and is the first general bound for the case of planes that depends also on k, except for the $O(nk^2)$ bound on the combined complexity of the first k levels [7]. Note that our new bound is smaller than this latter bound when $k = \Omega(n^{3/4})$.

The proof of Theorem 4.1 follows the previous proofs in [2, 8]. That is, it exploits a generalization of Lovász Lemma to three dimensions. We present here an improved version of this lemma, in arbitrary dimension, which leads to the improved bound of the theorem.

Let H be a collection of n hyperplanes in $\mathbb{R}^d$ in general position, and let $0 \le k \le n-d$. Let V_k denote the set of those vertices v of $\mathcal{A}(H)$ for which exactly k hyperplanes of H pass below v. For each $v \in V_k$, we denote by H_v the set of the d hyperplanes incident to v, and let R_v denote the closed region ('corridor') lying between the upper and lower envelopes of the hyperplanes of H_v.

Lemma 4.2 (Dual Lovász Lemma in $\mathbb{R}^d$) *For any $(d-2)$-flat f in $\mathbb{R}^d$, we have*

$$\left|\{v \in V_k \mid f \subset R_v\}\right| = O(k^{d-1}).$$

It will be more convenient to state and prove the primal version of this lemma. Fix a set S of n points in $\mathbb{R}^d$, in general position. A *k-set simplex* is a $(d-1)$-dimensional simplex spanned by d points of S with the property that its affine hull has precisely k points of S on one side of it.

Lemma 4.3 (Primal Lovász Lemma in $\mathbb{R}^d$) *Let S be a finite point set in $\mathbb{R}^d$. Then, for any line ℓ, the number of k-set simplices meeting ℓ is $O(k^{d-1})$.*

Proof: Note that this formulation of the lemma is independent of the choice of the coordinate system. Construct a coordinate system in which ℓ coincides with the x_d-axis. Dualize S to a system S^* of n hyperplanes, using the standard duality that maps a point $(a_1, \ldots, a_d)$ to the hyperplane $x_d = -a_1x_1 - a_2x_2 - \cdots - a_{d-1}x_{d-1} + a_d$, and a hyperplane $x_d = b_1x_1 + b_2x_2 + \cdots + b_{d-1}x_{d-1} + b_d$ to the point $(b_1, \ldots, b_d)$ (see, e.g., [9]); this duality preserves incidences and above-below relationships between points and hyperplanes (that is, a point p lies below, on, or above a hyperplane h if and only if the dual hyperplane p^* of p lies below, on, or above the point h^* dual to h). An application of such a duality also shows that this lemma and the preceding one are indeed dual versions of each other. It suffices to count the number of k-set simplices whose affine hulls have k points of S strictly below them. The remaining class of k-set simplices is handled by a symmetric argument.

The properties of the duality imply that the affine hull of a k-set simplex Δ as above is mapped into a vertex Δ^* of the arrangement of S^* which has precisely k hyperplanes below it (and d hyperplanes passing through it). Hence Δ^* is a vertex of the k-th level of $\mathcal{A}(S^*)$. Moreover, Δ meets the x_d-axis ℓ if and only if the horizontal hyperplane through Δ^* is contained in R_{Δ^*}, i.e., Δ^* is a local maximum of the k-th level of $\mathcal{A}(S^*)$.[1] Indeed, Δ meets ℓ if and only if every hyperplane that contains ℓ does not have all vertices of Δ on one side. The set of these hyperplanes is mapped by our duality to the set of all the points at infinity in horizontal directions. Hence Δ meets ℓ if and only if every point at infinity in a horizontal direction lies in R_{Δ^*}, which is equivalent to the condition that the horizontal hyperplane through Δ^* is contained in R_{Δ^*}, as asserted. As shown by Clarkson [6], the number of local extrema of the k-th level in an arrangement of hyperplanes in d-space is $O(k^{d-1})$, and this completes the proof of the lemma. □

Proof of Theorem 4.1: Dey and Edelsbrunner [8] have shown that if T is a collection of t triangles in 3-space, spanned by n points in general position, then there exists a line ℓ that crosses $\Omega(t^3/n^6)$ triangles of T. Specifically, consider the collection of the k-set triangles of an n-point set S. Lemma 4.3 implies that no line can cross more than $O(k^2)$ of these triangles. Combining this with the result of [8], we have $|V_k|^3/n^6 = O(k^2)$, and the bound follows. □

5 Arrangements of Triangles

Let $\mathcal{T} = \{\Delta_1, \ldots, \Delta_n\}$ be a collection of n triangles in 3-space in general position, and let $\mathcal{A}(\mathcal{T})$ denote the arrangement of $\mathcal{T}$. The k-th level of $\mathcal{A}(\mathcal{T})$ is defined, again, as the closure of the set of all points that lie in the union of the triangles and have exactly k triangles lying below them (that is, the relatively open vertical downward-directed ray

[1] The connection between local extrema of k-levels and Lovász Lemma was first observed by Clarkson, as briefly remarked in the introduction of [6].

emerging from such a point intersects exactly k triangles). As in the case of segments, the k-th level is not necessarily connected, and may have jump discontinuities at points that lie vertically above or on some triangle edge. The complexity of the level, regarded as a polyhedral surface, is the number of its vertices, edges and faces. Assuming general position, this is clearly proportional to the number of vertices only, and we will focus on bounding the number of *inner* vertices, which are incident to three distinct triangles. Any other, 'outer' vertex of the level lies in the vertical plane H_e spanned by some triangle edge e. Moreover, if we intersect all the triangles with H_e, we get a collection of at most n segments, and the vertices of the k-th level of $\mathcal{A}(\mathcal{T})$ that lie in H_e are vertices of the k-th level of the 2-dimensional arrangement of these segments within H_e, where e itself is also included. By Theorem 3.1, the number of such vertices is $O(n^{3/2})$. Repeating this analysis for each triangle edge e, we conclude that the number of outer vertices of the level is $O(n^{5/2})$.

We bound the number of inner vertices using a variant of the dual version of Lovász Lemma in 3-space. The bound that we obtain is considerably weaker than the one given in Lemma 4.2, but is still nontrivial. The proof of this version of the lemma is also different and somewhat more involved.

Let v be an inner vertex of the k-th level, incident to three triangles Δ_1, Δ_2, Δ_3; v can be classified into three categories, depending on whether the k-th level in the neighborhood of v coincides with

(a) the lower envelope of Δ_1, Δ_2, Δ_3,

(b) the first level of the arrangement $\mathcal{A}(\{\Delta_1, \Delta_2, \Delta_3\})$, or

(c) the upper envelope of Δ_1, Δ_2, Δ_3.

Note that vertices of type (b) have the property that all six edges of $\mathcal{A}(\mathcal{T})$ incident to the vertex lie on the k-th level, whereas for vertices of type (a) or (c), only three of these edges lie on the level, one edge on each segment of intersection of two of the triangles Δ_1, Δ_2, Δ_3.

For each inner vertex v of the k-th level of type (a) or (c), let R_v be the closed region enclosed between the upper envelope and the lower envelope of the three planes containing the three triangles incident to v; see Figure 7 for a cross-section of such an R_v. We have the following weaker version of Lovász lemma:

Lemma 5.1 *Any line in $\mathbb{R}^3$ is fully contained in at most $O(n^{5/2})$ regions R_v of vertices of type (a) and (c).*

Proof: Let ℓ_1 be a line in $\mathbb{R}^3$, and let H be the vertical plane containing ℓ_1. For a triangle $\Delta \in \mathcal{T}$, let π_Δ be the plane containing Δ and $\sigma_\Delta = \pi_\Delta \cap H$. Let $\mathcal{A}_H$ be the arrangement in H of the lines $\{\sigma_\Delta \mid \Delta \in \mathcal{T}\}$. Let ℓ_0 be a line contained in H, parallel to ℓ_1, and lying below all vertices of $\mathcal{A}_H$. It is easily checked that no region R_v contains ℓ_0. We will move a line ℓ within H upwards, parallel to itself, from the position when it coincides with ℓ_0 until it coincides with ℓ_1. We estimate the change in the number of regions R_v that contain ℓ as it moves. Summing these changes yields the bound on the desired quantity for ℓ_1.

The set of regions R_v that fully contain ℓ can change only when ℓ passes through a vertex of $\mathcal{A}_H$. Clearly, the vertex $\chi = \sigma_{\Delta_1} \cap \sigma_{\Delta_2}$ has to be such that there is an inner type-(a) or type-(c) vertex v in $\mathcal{A}$ incident to Δ_1 and Δ_2. Under these assumptions, for ℓ to become newly contained in a region R_v, or to stop being contained in R_v, as it sweeps past such a vertex χ, it is necessary and sufficient that the slope of ℓ lie between the slopes of σ_{Δ_1} and σ_{Δ_2}; see Figure 7(a). Let χ be such a vertex (where this latter condition also holds). Put $\lambda = \pi_{\Delta_1} \cap \pi_{\Delta_2}$. Let $s \subset \lambda$ be the segment $\Delta_1 \cap \Delta_2$. For all regions R_v that either start or stop containing ℓ as it sweeps over χ, v is contained in s, so it suffices to concentrate only on such regions R_v.

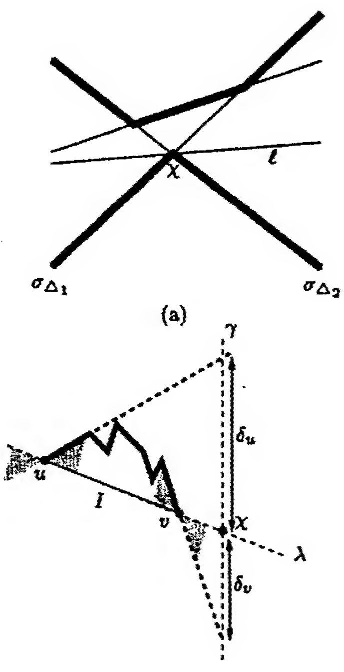

Figure 7: (a) Cross section of a region R_v in H; the line ℓ just becomes contained in R_v; (b) cross section of $\mathcal{A}(\mathcal{T})$ by π; the intersections of R_v, R_u with π are shaded near the respective vertices.

We mark on s all the inner vertices of the k-th level of $\mathcal{A}(\mathcal{T})$ of types (a) and (c), and consider the set of maximal subintervals of s *not* contained in the k-th level. Each such subinterval I is delimited by two points u, v, each of which is an inner vertex of the level of type (a) or (c) (it cannot be a vertex of type (b) because all six edges incident to a type-(b) vertex lie on the k-th level), a point of jump discontinuity of the level, or an endpoint of s. Let q_s be the number of jump discontinuities of the k-th level along s. Note that each such discontinuity is an outer vertex of the k-th level. If an inner vertex v is an endpoint of an interval along s whose other endpoint v' is either a jump discontinuity or an endpoint of s, we charge v to v'. The number of such inner vertices (and therefore the change in the number of regions that contain ℓ, corresponding to such vertices) is at most $q_s + 2$.

Next, consider an interval I, both of whose endpoints are inner vertices, say u and v. Consider the vertical plane π containing λ, and the cross-section of $\mathcal{A}(\mathcal{T})$ within π (refer to Figure 7(b)). Clearly, the k-th level of this cross-section is contained in the k-th level of $\mathcal{A}(\mathcal{T})$, so it either lies fully above I or fully below I. In the former case both u and v are of type (c), and in the latter case they are both of type (a). Let γ be the vertical line $H \cap \pi$, and let $\delta_u = R_u \cap \gamma$ and $\delta_v = R_v \cap \gamma$. If $\chi \notin I$, then it is easily checked that

δ_u and δ_v lie on opposite sides of χ along γ and thus are disjoint except at their common endpoint χ. This fact, and our assumptions that the slope of ℓ is between the slopes of $\sigma_u = \Delta_u \cap H$ and $\sigma_v = \Delta_v \cap H$, imply that one of R_u, R_v must be added, and the other one removed, from the set of regions containing f, as ℓ sweeps over χ. Hence, as ℓ sweeps over χ, R_u and R_v "cancel" out each other, in terms of containment of ℓ.

To summarize, we have shown that as ℓ passes through χ, the change in the number of regions R_v containing ℓ is at most $4+q_s$. This implies that the number of regions R_v that contain ℓ in its final position ℓ_1 is at most $\sum_s(4+q_s)$, where the sum is over all $O(n^2)$ intersection segments between pairs of triangles in $\mathcal{T}$. Since the number of outer vertices on the k-th level is $O(n^{5/2})$, as argued above, and each is counted at most three times, $\sum_s q_s = O(n^{5/2})$. The number of regions containing ℓ is thus $O(n^2)+O(n^{5/2}) = O(n^{5/2})$, as asserted.

What if ℓ_1 actually passes through a vertex $\chi = \sigma_{\Delta_i} \cap \sigma_{\Delta_j}$ of $\mathcal{A}_H$? Then the cancellation does not occur, which adds fewer than n regions R_v that can contain ℓ—each such region corresponds to some vertex of $\mathcal{A}(\mathcal{T})$ on the segment $\Delta_i \cap \Delta_j$. □

Theorem 5.2 *The complexity of any single level in an arrangement of n triangles in 3-space is $O(n^{17/6})$.*

Proof: Lemma 5.1 implies that no line ℓ is contained in more than $O(n^{5/2})$ regions R_v. Passing to the dual space, we obtain the following equivalent formulation, similar to the case of planes: The planes containing the triangles in $\mathcal{T}$ are mapped to a set of n points. Each inner vertex v of the k-th level is mapped to a triangle spanned by the three points dual to the planes containing the triangles incident to v. The line ℓ is mapped to another line ℓ^*, and ℓ is contained in R_v if and only if ℓ^* crosses the triangle dual to v. We now have a system of X triangles in 3-space, spanned by a total of n points, where X is the number of inner vertices of the k-th level of types (a) and (c). As in the proof of Theorem 4.1, there exists a line that crosses at least $\Omega(X^3/n^6)$ such triangles [8]. On the other hand, by Lemma 5.1, this number is at most $O(n^{5/2})$. Combining these two inequalities yields $X = O(n^{17/6})$. We still need to bound the number of vertices of type (b). However, these vertices are vertices of type (a) of the $(k-1)$-st level, so, repeating the above analysis for this level, we obtain the bound asserted in the theorem. □

Remark: An open problem is to extend Lemmas 4.2 and 5.1 to the respective cases of pseudo-hyperplanes and pseudo-triangles, under appropriate definitions of these objects, and then to extend the proof of Theorem 5.2 to these cases. Note that there are two different problems to address: One is to extend Lovász Lemma, and the other calls for a dual and more general version of the analysis technique of [8] (that yields a line that stabs many triangles).

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